abstract  Empirically we present some novel entailment patterns in English, and begin to characterize them semantically. Despite reliable judgments of entailment these patterns have gone largely unnoticed in work on philosophical logic and natural language semantics, possibly because many of the sentence pairs instantiating a pattern naturally invoke quantifier types not studied in standard logic, indeed not even definable in first order logic. Also in one large class of cases the judgments of entailment rely on relations between pairs of quantifiers, so it is more natural to consider each pair as a binary quantifier rather than an instance of iterated unary quantifiers.

This paper is organized as follows: §1 presents two novel entailment patterns and §3 the semantic generalizations they instantiate. §2 provides some formal background. We conclude with some additional, unstudied, entailment patterns, offered as a stimulus to further research.

§1. Pattern 1 In each (a,b) pair in (1) – (3), each sentence entails the other:

(1) a. More than 90% of the candidates have at least one sibling
   b. Less than 10% of the candidates have no siblings at all
      a. At least two thirds of the students answered no question correctly
      b. At most a third of the students answered at least one question correctly
      a. Exactly half the students read more plays than poems
      b. Exactly half the students read at least as many poems as plays

(2) a. Every student but John read at least as many plays as poems
   b. No student but John read fewer plays than poems
      a. All but finitely many English sentences contain an embedded sentence
      b. Just finitely many English sentences contain no embedded sentence
      a. All but a tenth of the students attended at least one demonstration this year
      b. A tenth of the students attended no demonstration this year

(3) a. Not every student read at least as many plays as poems
   b. Some student read more poems than plays
      a. Almost all stockbrokers read at least one financial paper at breakfast
      b. Hardly any stockbrokers read no financial paper at breakfast
      a. Each student answered at least two questions correctly on the exam
      b. No student answered fewer than two questions correctly on the exam
      a. Both John and Bill read at most two plays
      b. Neither John nor Bill read more than two plays

Our generalization for these cases concerns the relation between the pairs of Noun Phrases (NPs) in the (a,b) pairs, e.g. \(<\text{more than 90}\% \text{ of the candidates}, \text{ at least one sibling}>\) in (1a) and \(<\text{less than 10}\% \text{ of the candidates, no siblings at all}>\) in (1b). The generalization supports the logical
equivalence of the (a,b) pairs in (4) as well as the failure of equivalence in the (a,c) pair:

(4) a. Each witness told either John or Bill at least two lies
   b. No witness told both John and Bill fewer than two lies
      a. All but one judge awarded himself at least two prizes
      b. Exactly one judge awarded himself less than two prizes
      a. More than half the witnesses told every detective at least two lies
      b. Less than half the witnesses told even one detective fewer than two lies
      c. ≠ Less than half the witnesses told every detective at least two lies

Pattern 2 The (a,b) pairs in (5) are similarly mutually entailing. Their subject NPs are identical, and their Predicates are negations of each other. This pattern is unexpected in standard logic where it never happens that QxΦ and Qx¬Φ are logically equivalent, Q = ∨ or ∃.

(5) a. Between a third and two thirds of the students laughed at that joke
   b. Between a third and two thirds of the students didn't laugh at that joke
      a. Some but not all of the students laughed at that joke
      b. Some but not all of the students didn't laugh at that joke
      a. Either every student but John or else no student but John got an A on the exam
      b. Either every student but John or else no student but John didn't get an A on the exam
      a. Either all of the students or else none of them will get an A on the exam
      b. Either all of the students or else none of them will not get an A on the exam

§2 Background We interpret Ss like (6) relative to a model with universe E (arbitrarily chosen

(6) S
   NP
   Det₁ N
   P₁

Most poets daydream

and held fixed in what follows) as elements of the boolean lattice {0,1} of truth values (0 = False, 1 = True). P₁s, one place predicates, such as daydream, daydream in class, etc. and nouns N, like poet, cheerful poet, etc. denote properties, represented as subsets of the universe E. Initially we interpret NPs, such as John, most poets, more students than teachers,... as generalized quantifiers (GQs) — type ≺<1> functions, from properties to truth values. Noting interpretations in upper case, (6) is interpreted by (MOST POET)(DAYDREAM), the truth value that the MOST POET quantifier assigns to DAYDREAM.

The set GQE of generalized quantifiers over E, is [P(E) → {0,1}], P(E) the power set of E and in general [A → B] the set of functions from A into B. Det₁'s, one place Determiners, such as most, most but not all,... map properties to GQs and thus lie in [P(E) → GQE]. Det₂'s, such as more...than..., fewer...than..., and twice as many...as..., as in (7), map pairs of properties to GQs.

(7) a. More students than teachers attended the party
   b. Fewer plays than poems were on the reading list
c. Twice as many cats as dogs are in the garden

For example, ALL maps a property such as POET to that generalized quantifier ALL(POET), which maps a set like DAYDREAM to T iff POET ⊆ DAYDREAM. Some poets daydream is True iff the set of poets who daydream is non-empty, that is, POET ∩ DAYDREAM ≠ ∅. No poets daydream iff that set is empty, and Most poets daydream is True iff the poets that daydream outnumber those that don’t. More poets than reporters daydream is true iff the poets that daydream outnumber the reporters that daydream. Some further examples are:

(8) a. (LESS THAN n)(A)(B) = 1 iff |A ∩ B| < n
b. (EXACTLY n)(A)(B) = 1 iff |A ∩ B| = n
d. (ALL BUT n)(A)(B) = 1 iff |A − B| = n
d. BOTH(A)(B) = 1 iff |A| = 2 and A ⊆ B
e. NEITHER(A)(B) = 1 iff |A| = 2 and A ∩ B = ∅
f. (LESS THAN HALF)(A)(B) = 1 iff 2|A ∩ B| < |A|
g. (MORE THAN TWO THIRDS)(A)(B) = 1 iff 3|A ∩ B| > 2|A|
h. (FEWER A THAN B)(C) = 1 iff |A ∩ C| < |B ∩ C|
i. (n TIMES AS MANY A AS B)(C) = 1 iff |A ∩ C| = n|B ∩ C|

Interpreting NPs as generalized quantifiers as above is natural when they are subjects of predicates, but they also function as direct and indirect objects:

(9) a. Each student admires at least one teacher
b. No student gave more than one teacher an apple

In general (Keenan & Westerståhl1997) NPs combine with n+1-ary predicates to form n-ary ones and can be interpreted as maps from n+1-ary relations, elements of P(E^{n+1}), to n-ary ones. The case of basic concern to us is when n = 2, given explicitly below:

**Definition 1** For F a generalized quantifier and R a binary relation over E,

\[ F(R) = \{ x \in E | F(\{ y \in E | xRy \}) = 1 \} \]

So the value F assigns to a binary relation is decided by those it assigns to the unary ones (the subsets of E). In (10a) admires at least one teacher denotes the set of objects which stand in the ADMIRE relation to at least one teacher. This is the set that (AT LEAST ONE TEACHER) assigns to ADMIRE: the set of x such that (AT LEAST ONE TEACHER) is true of the set of objects x admires. Of course there are many more maps from binary to unary relations (|E| > 1) than from unary to 0-ary ones (the truth values). Not all satisfy the equation in Def 1. Those that do (Keenan 1989) are just those satisfying the Accusative Extensions Condition:

(10) **AEC** (Accusative Extension Condition): for all a,b ∈ E, all R,S ⊆ E×E,

\[ a \in F(R) \iff b \in F(S) \text{ whenever } \{ y | aRy \} = \{ y | bSy \} \]

The NPs satisfying the AEC may be complex. For example, most of John’s students satisfies the AEC since if the objects Bill praised are exactly those that Mary criticized then Bill praised most of John’s students and Mary criticized most of John’s students must have the same truth value.

**Boolean structure** Expressions of a given category denote in a set with a boolean structure. For n place predicates the denotation set P(E^n) is a power set, the boolean relation is subset, and the boolean operations are (arbitrary) intersection, union and set theoretical complement. This holds
for S = P₀ as well, taking 0 (False) as the empty set ∅ and 1 (True) as \{∅\}; and in any case the
boolean meet (\∧\), join (\∨\) and complement (¬) operations are given by the standard truth tables
for conjunction, disjunction and negation respectively. And in general when A and B are
boolean lattices [A \rightarrow B] inherits boolean structure pointwise from B: f ≤ g iff for all a ∈ A, f(a) ≤ g(a); (f\∧ g)(a) = f(a)\∧ g(a) and (¬f)(a) = ¬(f(a)). The judgments of logical equivalence below support the correctness of pointwise interpretations for boolean compounds of NPs and Dets,
where and, or, and not denote \∧, \∨, and ¬ in whatever lattice the expressions they combine with
denote:

\[(11)\]
\[
a. \text{Every boy and some girl wept} \equiv \text{Every boy wept and some girl wept} \\
b. \text{Not a creature was stirring} \equiv \text{It is not the case that a creature was stirring} \\
\text{Not more than two boys wept} \equiv \text{It is not the case that more than two boys wept} \\
c. \text{(Most but (not all)) birds fly} \equiv \text{Most birds but (not all) birds fly} \\
\text{Most birds but (not (all birds)) fly} \equiv \text{Most birds fly but it is not the case that all birds fly}
\]

We are particularly interested in the expression of boolean complements of NPs. When an NP
admits of overt negation with not the two stand in a complement relation, as in:

\[(12)\]
\[
\text{every student COMPLEMENT not every student} \\
\text{not a (single) student} \\
\text{not both John and Bill} \\
\text{not more than two boys} \\
\text{not more than half the students} \\
\text{not more students than teachers}
\]

But certain NPs don’t negate naturally with not. Not every student attended is natural, but *Not
the student attended, *Not John attended are not (as indicated by the asterisk). And often NPs
which negate in subject position are awkward in object position. Thus while pointwise negation
tells us that (13a,b,c) are logically equivalent, (13b) is of marginal grammaticality:

\[(13)\]
\[
a. \text{He praised every student and/but not a single teacher} \\
b. ??\text{He praised every student and/but praised not a single teacher} \\
c. \text{He praised every student and/but didn’t praise a single teacher}
\]

But many pairs of NPs stand in the complement relation (meaning that each denotes the boolean
complement of the other) even though they are not built with overt negation:

\[(14)\]
\[
\text{more than ten boys COMPLEMENT at most ten boys} \\
\text{at least half the plays} \equiv \text{less than half the plays} \\
\text{either John or Bill} \equiv \text{neither John nor Bill} \\
\text{some student} \equiv \text{no student} \\
\text{at least two plays} \equiv \text{less than two plays} \\
\text{one or more plays} \equiv \text{no plays} \\
\text{(at least) as many poems as plays} \equiv \text{fewer poems than plays} \\
\text{(at least) as many poems as plays} \equiv \text{more poems than plays} \\
\text{more boys than girls} \equiv \text{at most as many boys as girls} \\
\text{exactly half the boys} \equiv \text{either less or more than half the boys} \\
\text{at least ten students} \equiv \text{fewer than ten students}
\]

The pairs below are logically equivalent, complements understood pointwise: (¬F)(R) =

\(\neg(F(R))\).
Most of the equivalences in (15) are not standard textbook examples, as they involve GQs that lie over the first order horizon, such as the proportionality quantifiers (fractions and percentage ones) (Barwise and Cooper (1981) and Westerståhl 1989) and cardinal comparatives:

§3 Pattern 1 is characterized partly in terms of the complement relation and partly in terms of a more novel boolean relation I call postcomplement (called contradual in Zwarts 1996):

Definition 2 For F,G generalized quantifiers, F postcomplement G iff for all properties p,

\[ F(p) = G(\neg p) \]

The NPs on the left in (16) stand in the postcomplement relation to those on the right. For example, from Every student passed we infer No student didn’t pass and conversely, etc.

As postcomplement plays a fundamental role in what follows we note here some of its properties. First, if F is a postcomplement of G then its value at each property is determined, whence:
PROPOSITION 1 Each GQ $G$ has a unique postcomplement, noted $G\tilde{-}$, so $\tilde{-}$ will be regarded as a function, the postcomplement function, mapping GQs to GQs. 

Secondly, postcomplement inherits some properties from the complement operation:

PROPOSITION 2 a. postcomplement is symmetric: $F = G\tilde{-}$ iff $G = F\tilde{-}$.

proof $F = G\tilde{-}$ iff for every property $p$, $F(p) = G(\neg p)$
iff for every $p$, $F(\neg p) = G(\neg\neg p)$, $= G(p)$

b. postcomplement is self-invertible: $(G\tilde{-})\tilde{\tilde{-}} = G$.

proof $(G\tilde{-})\tilde{\tilde{-}}(p) = (G\tilde{-})(\neg p) = G(\neg\neg p) = G(p)$, whence $(G\tilde{-})\tilde{\tilde{-}} = G$

Characterizing Pattern 1 The subject NPs in the pairs in (17) are postcomplements of and their P1s complements, so we infer that the (a,b) pairs are logically equivalent. And by the symmetry of postcomplement the subject NPs of each pair can be interchanged preserving entailment.

(17) a. More than 90% of the candidates didn't answer question 6 correctly
b. Less than 10% of the candidates answered question 6 correctly

a. Exactly three out of ten students smoke
b. Exactly seven out of ten students don't smoke

a. All but a tenth of John's students got scholarships
b. A tenth of John's students didn't get scholarships

a. Both John and Bill didn’t sign the letter
b. Neither John nor Bill signed the letter

This is the pattern exhibited by the pairs in (1) except that the complement relation among the P1s is derived by using complement object NPs with the same P2, not by overtly negating one of the P1s. To express this succinctly we define:

Definition 3 For $F,G$ generalized quantifiers, the type $<2>$ quantifier (mapping $n+2$-ary relations to $n$-ary ones) induced by $(F,G)$ is $(F\circ G)$, defined by:

$$(F\circ G)(R) = F(G(R)).$$

PROPOSITION 3 For generalized quantifiers $F,G$, $(F\circ G) = (F\tilde{-}(\neg\circ\neg G))$

Proof $$(F\tilde{-}(\neg\circ\neg G))(R) = (F\tilde{-}((\neg\circ\neg G)(R)))$$
= $(F\tilde{-})(\neg(\circ\neg G(R)))$ Pointwise $\neg$
= $F\tilde{-}((\circ\neg G(R)))$ Def $\circ$
= $F(G(R))$ Double complements

Generalization 1 The logical equivalences in (1) – (4) are all of the form $(F\circ G)(R) = (F\tilde{-}(\circ\neg G))(R)$

The (a,b) pairs in (1) – (4), as indicated below, are built from the same transitive verb using subject NPs that are postcomplements and object NPs that are, so they instantiate Gen 1.

(1) a. [More than 90% of the candidates] have [at least one sibling]
b. [Less than 10% of the candidates] have [no siblings at all]

**more on postcomplement** The judgment that two NPs are postcomplements may be good even if judgments of the denotations of the NPs are not. That (18a,b) are mutually entailing is clear and depends on the judgment that *almost all* As and *hardly any* As are postcomplements, though the truth conditions for *Almost all / Hardly any* As are Bs are unclear. Similarly for (19a,b):

(18) a. Almost all stockbrokers have a martini after work
   b. Hardly any stockbrokers don’t have a martini after work

(19) a. A large proportion of the students oppose a tuition increase
   b. A small proportion of the students don’t oppose a tuition increase

Second, while NPs of the form X and *not* X denote complement GQs, there appears to be no word or morpheme in English which attaches to an NP to form one understood as its postcomplement. So postcomplement is less regularly coded in English, and likely in languages generally, than complement. But we do find a regularity among the Dets: when the paired Dets below combine with the same N the resulting NPs are postcomplements:

(20) a. More than n/m POSTCOMPLEMENT Less than 1 – n/m
   b. Exactly n/m (0 ≤ n ≤ m; 0 < m) Exactly 1 – n/m
   c. At most n/m At least 1 – n/m

Similarly for *more than* (exactly, *at most*) n% versus *less than* (exactly, *at least*) (100 – n)% and *more than* (exactly, *at most*) n of the m versus *less than* (exactly, *at least*) (m – n) of the m. What we need to study here then are the numerical specifiers *more than*, *exactly*, and *at least/least*. While these observations need to be generalized, it is already encouraging to find logical regularities among the proportionality quantifiers, as they are poorly understood.

Third, let us note two ways in which postcomplement differs from complement. First, it is not a general boolean operation, it only makes sense when speaking of functions whose domains have a boolean structure, one with a complement operation. It does correspond to the *contrary* and the *subcontrary* relations in the classical Aristotelian square of opposition. Here the diagonal

(21) ALL NO

SOME NOT ALL

relations, ALL – NOT ALL and SOME – NO are called *contradictories* and correspond to the complement function, present in all boolean lattices. The top horizontal relation, ALL – NO is the *contrary* relation, and the lower one the *subcontrary* relation. Both are postcomplement relations. (The two vertical relations ALL – SOME and NO – NOT ALL are called *subalternates* and correspond to what we shortly call *duals*). Secondly,

**PROPOSITION 4** Postcomplement is an automorphism of GQE.

a. Like ¬, – is self invertible: (–)² = id, and thus is bijective.

b. Unlike ¬, – preserves meets: \((F \lor G)(\neg p) = (F \lor G)(\neg p)\) \(\quad\) Def –

\(\quad\) = \(F(\neg p) \lor G(\neg p)\) \(\quad\) Pointwise meets

\(\quad\) = \((F)(\neg p) \lor (G)(\neg p)\) \(\quad\) Def –
\[ (F\land G)-(p) = (F-(G-))(p) \]

Pointwise meets

Whence \((F\land G)-( = (F-(G-)).\) Replacing \(\land\) by \(\lor\) above shows that \(-\) preserves \(\lor\). \(\Box\)

c. \(-\) preserves \(\neg\) (as does \(\neg\) itself): \((-F)-( = \neg(F-(), and we may unambiguously write \(\neg F-(.\)

\[
((-F)-(p) = \neg(F(-p)) \quad \text{Def } - \\
= \neg((F-)-(p)) \quad \text{Pointwise complement} \\
= (\neg(F-))(p) \quad \text{Def } - \\
= (\neg(F-))(p) \quad \text{Pointwise complement} \quad \Box
\]

So postcomplement is an automorphism of \(GQ_E\). By contrast complement in any boolean lattice \((B,\leq)\) is an isomorphism onto the dual lattice, \((B,\geq), as per the De Morgan laws.

**Generalizing with NP duals**

The logical equivalences in (22) crucially involve three NPs:

(22) a. Each counselor told both John and Bill at least three stories
    b. No counselor told either John or Bill fewer than three stories

    a. Not every witness told every detective two or more lies
    b. At least one witness told some detective fewer than two lies

    a. More than half the witnesses told every detective at least two lies
    b. Less than half the witnesses told any detective fewer than two lies

    a. All but one witness told more than half the detectives at least one lie
    b. Exactly one witness told at least half the detectives no lie at all

In these examples, as in (1) \(- (4), the initial NPs are postcomplements, and the final NPs are complements. The relation between the corresponding interior NPs is forced by:

**Theorem 5: Facing Negations** (Keenan 1993): For \(F,G,F',G'\) generalized quantifiers, if neither \(F\) nor \(G\) are trivial (= constant) then \(F\circ G = F'\circ G'\) iff either \(F = F'\) and \(G = G'\) (a trivial case) or \(F\) and \(F'\) are postcomplements and \(G\) and \(G'\) are complements. \(\Box\)

Now, representing the (a,b) equivalences in (22) as in (23), we ask what the logical relation

(23) \((F\circ H\circ G)(R) = (F-\circ H'\circ -G)(R).\)

between \(H\) and \(H'\) must be for this equality to hold for all \(R.\) The answer is given by Corollary 7: in our examples we need two facing negations, so \(H'\) must be \(\neg H-\), the dual of \(H\), often noted \(H^d\). For example, every pupil and some pupil are duals. If some pupil has \(p\) then it is not the case that every pupil doesn't have \(p\), and conversely. And clearly, (24), equality holds in (23) replacing \(H'\) by \(H^d\). Corollary 7 below draws on Lemma 6.

(24) \(F-\circ H-\circ -G = F-\circ ((\neg H-\circ -G) \circ \text{ is associative} \\
= F-\circ (\neg H)\circ G \quad \text{Facing Negations} \\
= (F-\circ \neg H)\circ G \circ \text{ is associative} \\
= (F\circ H)\circ G \quad \text{Facing Negations} \quad \Box
\)

**Lemma 6** for \(F,G\) generalized quantifiers

1. If \(F\) is non-trivial (= not constant) then \(\neg F, F-,\) and \(F^d\) are also non-trivial, and
2. \((\neg F \circ G) = \neg(F \circ G)\) and \((F \circ G -) = (F \circ G) -\), and
3. If \(F\) and \(G\) are non-trivial so is \(F \circ G\).

Proof of 3: We may assume \(G(\varnothing) = 0\), otherwise use \((F - \circ \neg G) = F \circ G\); one of \(G\) and \(\neg G\) maps \(\varnothing\) to 0. Since \(G\) is non-trivial let \(G(s) = 1\). So \(s \neq \varnothing\). Since \(F\) is non-trivial let \(F(p) \neq F(q)\), \(p \neq q\). Then \((F \circ G)(p \times s) = F(G(p \times s)) = F(\{x | G(\{y | (x, y) \in p \times s\})/1\}) = F(p)\). And \(q \times s \neq p \times s\) and \((F \circ G)(q \times s) = F(G(q \times s)) = F(\{x | G(\{y | (x, y) \in q \times s\})/1\}) = F(q) \neq F(p)\), so \(F \circ G\) is non-trivial. □

Corollary 7 For \(F, G, H\) non-trivial GQs, \((F \circ H \circ G) = (F' \circ H' \circ G')\) iff one of the four not fully exclusive options below obtains (proof by judicious application of Facing Negations):

<table>
<thead>
<tr>
<th>Options</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
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<tbody>
<tr>
<td>(F')</td>
<td>(F)</td>
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<td>(F-)</td>
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<tr>
<td>(H')</td>
<td>(H)</td>
<td>(H-)</td>
<td>(\neg H)</td>
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<tr>
<td>(G')</td>
<td>(G)</td>
<td>(\neg G)</td>
<td>(G)</td>
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Option (4) is the only one that applies in (22), where \(F' \neq F\) and \(G' \neq G\), forcing \(H'\) to be the dual of \(H\). More duals are given in (25). Prop 8 helps justify including certain pairs given others.

PROPOSITION 8 Duality a. is symmetric \(F = G^d\) iff \(G = F^d\), and
b. preserves complement \(F = G^d\) iff \(\neg F = (\neg G)^d\), and
c. preserves postcomplement \(F = G^d\) iff \(F- = (G-)^d\), and
d. is self-inverting \(F^{dd} = F\)

proof: a. \(F = G^d\) iff \(F = \neg G-\), iff \(F- = (\neg G-)\), iff \(\neg F- = \neg G\), iff \(\neg F- = G\), iff \(G = F^d\). (b), (c) and (d) are similarly easy. □

(25) both John and Bill \hspace{1cm} DUAL \hspace{1cm} either John or Bill
more than seven out of ten sailors \hspace{1cm} at least three out of ten sailors
at most 70% of the students \hspace{1cm} more than 30% of the students
all but at most two students \hspace{1cm} not every student
no student \hspace{1cm} at least 6 of the 15 students
more than 9 of the 15 students \hspace{1cm} at most two out of three babies
less than one baby in three \hspace{1cm} at least one of the two students
both students \hspace{1cm} neither student
not both of the students

One verifies that the intermediate NPs in (22) are duals, instantiating (24). Equally one verifies the empirically somewhat surprising fact that we cannot simply hold the verb plus its immediate postverbal NP fixed, treating it as a complex two place predicate. For example (26b) does not entail (26a). If each counselor tells his favorite camper four stories but fails to tell any to the other campers then (26b) is true and (26a) is not. But note that Option (4) enables us to predict that we can hold the immediate postverbal NP constant when it is self dual, as in (27):

(27) a. Both counselors told John at least three stories
b. Neither counselor told John fewer than three stories
a. Each counselor told more than ten of the twenty-one campers at least two stories
b. No counselor told more than ten of the twenty-one campers fewer than two stories

a. All but one judge awarded himself at least one prize
b. Exactly one judge awarded himself no prize at all

Only the proper nouns have been previously noted as self dual (Barwise and Cooper, 1981). Clearly John has p iff it is not the case that he has not-p. We interpret proper nouns as *individuals*, where for each b ∈ E, I_b, or the *individual generated by b*, is that GQ sending a subset p of E to 1 iff b ∈ p. One verifies directly that I_b(p) = ¬I_b(¬p), so I_b is self dual. But the next example in (27) suggests a systematic way of building self dual NPs. To this end we observe

(28) If a subset p of E is finite with odd cardinality then the GQs below are self dual:

a. (MORE THAN (|p| − 1)/2)(p)  a’. (MORE THAN HALF THE)(p)
b. (AT MOST (|p| − 1)/2)(p)  b’. (AT MOST HALF THE)(p)

So *more than two pupils* denotes a self dual quantifier when |PUPIL| = 5. By way of argument, given this assumption we observe that (29a,b) always have opposite truth values, so the negation of (29b) always has the same value as (29a), whence (29a) is self dual:

(29) a. more than two pupils passed the exam
   b. more than two pupils didn’t pass the exam

If (29a) is true then |PUPIL ∩ PASS| = 3, 4, or 5, so |PUPIL ∩ ¬PASS| = 2, 1, or 0, whence *more than two pupils didn’t pass* is false. If (29a) is false than |PUPIL ∩ PASS| = 2,1,or 0, so |PUPIL ∩ ¬PASS| = 3, 4, or 5. So *more than two pupils didn’t pass* is true. So in all cases (29a,b) have different truth values, as was to be shown.

**Characterizing the self dual quantifiers** A self dual quantifier is given by a two bloc partition of P(E), one bloc being the set of complements of the other. Given |E| = n both these blocs must have cardinality 2^n - 1. Since a self dual F assigns opposite values to a property and its complement, we can define such an F by giving its values freely on just one of the blocs. Thus the number of self dual quantifiers is 2^k, where k = 2^n - 1. Despite these “power set” figures though, the set of self dual quantifiers is not a boolean subalgebra of GQE. It does not contain the boolean 0 or 1, and is not closed under meet or join (but is closed under complement).

One does see easily though why proper nouns are self dual: the sets an individual I_b maps to 1 are just the supersets of {b} and so form a principal ultrafilter, including for each p ⊆ E, either p or ¬p but never both. For E infinite the homomorphisms induced by the non-principal ultrafilters are also self dual, but we know of no way to express them. Curiously though the “odd oriented” GQs in (28) are in some ways finite analogues of non-principal filters: they pick out the upper half of the Hasse diagram of P(E). For example suppose |E| = 5. Then MORE THAN 2 ENTITIES maps a subset B of E to 1 iff |B| ≥ 3 and so is true of the subsets of cardinality 3, 4, and 5, and false of their complements, of size 2, 1, and 0. Its complement, AT MOST 2 ENTITIES, maps each B of size 2, 1, or 0 to true, and their complements, of size 3, 4, and 5, to false.

More challenging here is to characterize the self dual *invariant* GQs – those invariant under the permutations π of E. Each such π extends to P(E) by setting π(p) = {π(x)|x ∈ p}; for F a GQ, π(F) is that GQ mapping each π(p) to F(p). The subsets of P(E) that are *permutation
invariant (PI), that is, fixed by all such \( \pi \), are just \( \emptyset \) and \( E \); the PI binary relations are \( \emptyset, E \times E \), ID, and \( \neg \text{ID} = \{ <x, y> \in E \times E | x \neq y \} \). For \( E \) finite, the PI GQs are the joins of the \( F_k \), \( 0 \leq k \leq |E| \):

\[
F_k(p) = 1 \text{ iff } |p| = k.
\]

(30) depends on the fact that there is a \( \pi \in \text{PERM}(E) \) such that \( \pi(p) = q \text{ iff } |p| = |q| \). (Were \( E \) infinite we would also have to require that \(|\neg p| = |\neg q|\). Since over an \( E \) with \( n \) elements there are \( n+1 \) cardinals \( \leq n \) we infer that there are \( 2^{n+1} \) GQs that are PI. At issue is which are self dual. To discover this the following proposition is useful:

**PROPOSITION 9** For any \( n \), a function \( F \) from \( n+1 \)-ary relations to \( n \)-ary ones is self dual iff \( F \) commutes with complement: \( F(\neg R) = \neg(F(R)) \).

**proof** Let \( F = \neg F^\partial \). Then \( F(\neg R) = \neg F^\partial(\neg R) = \neg F(\neg\neg R) = \neg(F(R)) \).

Let \( F \) commute with \( \neg \). Then \( (\neg F^\partial)(R) = (\neg F)(\neg R) = \neg(F(\neg R)) = \neg\neg(F(R)) = F(R) \), so \( F = \neg F^\partial \), whence \( F \) is self dual. \( \square \)

**Theorem 10**

1. If \( E \) is finite with \(|E| \) even or \( E \) is infinite then there are no PI self dual GQs.

**Proof**: Let \( E \) finite with \(|E| = 2n \). Suppose, leading to a contradiction, that \( F \) is a PI self dual GQ. Let \( K \) be an \( n \)-element subset of \( E \), whence \( \neg K \) also has \( n \) elements. Let \( \pi \) be a bijection transposing the elements of \( K \) with those of \( \neg K \). Then \( \pi \) is a permutation of \( E \) and since \( F \) is PI, \( F(K) = F(\pi K) = F(\neg K) = \neg(F(K)) \), so \( F \) fails to commute with \( \neg \) and thus is not self dual, the desired contradiction. Similarly if \( E \) is infinite find a subset \( K \) of \( E \) with \(|K| = |E| = |\neg K| \) and proceed as above.

2. If \( E \) is finite of cardinality \( 2k + 1 \) then there are \( 2^{k+1} \) PI self dual GQs. **Proof** For each function \( f : \{0, \ldots, k\} \to \{0,1\} \) define the GQ \( Q_f \) by:

\[
Q_f(p) = \begin{cases} 
  f(|p|) & \text{if } 0 \leq |p| \leq k \\
  \neg f(|p|) & \text{if } k < |p| \leq 2k + 1
\end{cases}
\]

Clearly \( Q_f \) commutes with complement, and so is self dual, since for each \( p \), \( p \) and \( \neg p \) satisfy different of the conditions above (a condition that fails when \(|E| \) is even), so \( f \) maps them to opposite truth values. And \( Q_f \) is PI since its value at any \( p \) is decided by \(|p| \). Since any way of assigning truth values to \( p \) and commuting with complement is given above these are all the PI self dual GQs. \( \square \)

By way of example, suppose \(|E| = 7 \). Then the following GQs are PI and self dual:

- EITHER 1 OR 2 OR 4 OR 7 ENTITIES; MORE THAN 3 ENTITIES;
- AT MOST THREE ENTITIES; MORE THAN HALF THE ENTITIES

**Caveat** Theorem 10 only applies to GQs considered as maps from \( P(E) \) into \( \{0,1\} \). The quantifier SELF from binary relations to sets below is self dual and PI, but not of the form above.

Finally consider self dual reflexives, like *himself* in (27). *Himself* laughs in standard English, and more deeply, it does not denote an accusative extension per Definition 1 of any map from \( P(E) \) into \( \{0,1\} \). But *himself*, interpreted
as SELF from binary relations to unary ones, is self dual, Props 11 and 12.

**Definition 4**

\[ \text{SELF}(R) = \{ x \in E | (x, x) \in R \} \]

**Proposition 11** (Keenan 1989) No \( F \in \{ \mathcal{P}(E) \to \{0,1\} \} \) extends per Definition 1 to binary relations \( R \) over \( E \) such that \( F(R) = \text{SELF}(R). \) \( \Box \)

When \( |E| > 1 \) SELF fails the AEC: It may be that Joe praised just the people Max criticized but that *Joe praised himself* is false and *Max criticized himself* is true. (Suppose that Joe praised just Sam, Sue, and Max, and that those are just the people Max criticized).

**Proposition 12** SELF commutes with complement, and is in fact a complete homomorphism.

*Proof* Re \( \neg \), an element \( a \in \text{SELF}(\neg R) \) iff \( (a,a) \in \neg R \), iff \( (a,a) \in R \), iff \( a \in \neg \text{SELF}(R) \), iff \( a \in \neg \text{SELF}(R) \), so \( \text{SELF}(\neg R) = \neg (\text{SELF}(R)). \) We leave preserving \( \land \) and \( \lor \) to the reader, noting simply the pre-theoretical judgments of equivalence:

\[
\begin{align*}
(31) \quad & a. \text{ Al both praised and criticized himself} \equiv \text{ Al both praised himself and criticized himself} \\
& b. \text{ Al either praised or criticized himself} \equiv \text{ Al either praised himself or criticized himself}
\end{align*}
\]

A generalization manqué? The \( (a,b) \) pairs in (32) are mutually entailing:

\[
\begin{align*}
(32) \quad & a. \text{ Every student in the class answered the same questions on the exam} \\
& b. \text{ No two students in the class answered different questions on the exam} \\
& \quad a. \text{ Different people like different things} \\
& \quad b. \text{ No two people like exactly the same things}
\end{align*}
\]

Even the first example cannot be another instance of Facing Negations as Keenan (1992) shows that the type \( <2> \) function \( F_{A,B} \) mapping a binary relation \( R \) to 1 iff each two \( \alpha \in A \) bear \( R \) to a different subset of \( B \) is not \( F \circ G \) for any GQs \( F,G \). Similarly for the Different-Different quantifier in the second case. So Facing Negations should be generalized.

**Pattern 2** recall concerns quantifiers that take the same value at a property and its complement:

\[
\begin{align*}
(33) \quad & a. \text{ Exactly half the students came to the party} \\
& b. \text{ Exactly half the students didn't come to the party} \\
& \quad a. \text{ Between a third and two thirds of the students laughed at that joke} \\
& \quad b. \text{ Between a third and two thirds of the students didn't laugh at that joke}
\end{align*}
\]

Since the GQs denoted by these subject NPs map each \( p \) and its complement \( \neg p \) to the same value they must be their own postcomplements. So the postcomplement operator has fixed points, in distinction to boolean complement, which does not (if \( |B| > 1 \)). For the record:

**Proposition 13**

1. \( \neg \) has fixed points, and
2. For all GQs \( F, F = F\neg \iff \text{ for all properties } p, F(p) = F(\neg p) \) (proof omitted). \( \Box \)

**Generalization 2** Pattern 2 consists of pairs of Ss interpreted as \( F(p) \) and \( F(\neg p) \), where \( F = F\neg \).
It might seem that the set \( \text{FIX}(\neg) \) of GQs fixed by \( \neg \) is small: replacing \textit{half} by other proportions in (33) loses this property. \textit{Exactly two thirds of the boys came to the party} does not entail \textit{Exactly two thirds didn’t come}. Rather it entails \textit{Exactly one third didn’t come} since \textit{exactly two thirds of the boys} and \textit{exactly one third of the boys} are postcomplements. But more are fixed than meet the eye. Two trivial cases are 0, which maps all properties to 0, and 1, which maps all properties to 1. But several other cases are not trivial:

**Proposition 14** GQs of the form \( F \land F\neg \) and those of the form \( F \lor F\neg \) are fixed by \( \neg \).

**Proof** Clearly \( (F \land F\neg)\neg = F\neg \land ((F\neg)\neg) = F\neg \land F = F \land F\neg \). Replacing \( \land \) everywhere by \( \lor \) shows that \( (F \lor F\neg) \) is fixed by \( \neg \). \( \Box \)

Here are some NPs that denote GQs of the forms in Prop 14:

(34) a. Some but not all students
    b. More than a third and less than two thirds of the students
    c. Between a third and two thirds of the students
    d. At least 10\% but not more than 90\% of the students
    e. Exactly 50\% of the students (exactly 50 = at least 50 and not more than 50)
    f. Either just one student or else all but one student
    g. Either every student but John or else no student but John
    h. Either both John and Bill or else neither John nor Bill
    i. Either less than a third or else more than two thirds of the boys
    j. Either less than 10\% or else more than 90\% of the students

Consider (35a). If some but not all students have \( p \) then some do, so not all have \( \neg p \); and since not all have \( p \) then some must have \( \neg p \), hence some but not all students have \( \neg p \).

**Proposition 15** If \( F \) is fixed by \( \neg \) then so are \( \neg F \) and \( F\neg \). If \( F \) and \( G \) are both fixed then so are \( F \land G \) and \( F \lor G \). (The easy proofs are omitted).

**Proposition 16** The following are fixed by \( \neg \) for any \( F \) (proofs omitted).

a. \( \neg F \lor \neg F\neg \) b. \( \neg F \land \neg F\neg \) c. \( (F \land F\neg) \lor (\neg F \land \neg F\neg) \) d. \( (F \land \neg F\neg) \lor (F\neg \land \neg F) \)

As an instance of Prop 15a we have \textit{all or none of the students}. And for subject NPs that are their own postcomplements we have pairs of equivalent Ss differing just in having object NPs that are complements of each other. E.g.

(35) a. Some but not all of the students read two or more plays
    b. Some but not all of the students read less than two plays

    a. Either just one student or else all but one student read a poem over the vacation
    b. Either just one student or else all but one student read no poems over the vacation

    a. Either all of the students or else none of them read as many plays as poems
    b. either all of the students or else none of them read fewer plays than poems

**Theorem 17: Characterizing \text{Fix}(\neg)\)** Given \( |E| = n \), we have that \( |\text{GQ}_E| = 2^k \), for \( k = 2^n \). But \( |\text{FIX}(\neg)| = 2^j \), for \( j = 2^{n+1} \). Thus over a 4 element universe there are \( 2^{16} = 65,536 \) generalized quantifiers, of which \( 2^8 = 256 \) are fixed by the postcomplement function.
proof sketch  Show that \(\text{FIX}(\neg)\) is a complete atomic boolean subalgebra of \(\text{GQE}\). The atoms are the functions \(h_p\) given by \(h_p(q) = T\) iff \(q = p\) or \(q = \neg p\). Clearly \(h_p = h_{\neg p}\) but that is the only non-trivial case, so the number of atoms divides by 2 the cardinality of \(\text{P}(E)\). So the elements of \(\text{FIX}(\neg)\) are just the joins of the \(h_p\), expressible as in (36) below. □

\[(36)\] Either every \(N\) and no non-\(N\)s or every non-\(N\) and no \(N\)s

Every student and no non-student holds just of the STUDENT property, so the NP in (36) holds just of the denotation of \(N\) and its complement. So in principle every element of \(\text{FIX}(\neg)\) over a finite universe can be expressed as a disjunction of NPs of the form in (36).

As earlier let us consider which invariant GQs fix \(\neg\). A boolean perspective leads to an answer: The invariant elements of \(\text{GQE}\) constitute a complete and thus atomic subalgebra of \(\text{GQE}\), as does \(\text{FIX}(\neg)\). Hence their intersection is complete and thus atomic. And one computes the atoms of this algebra to be:

**Theorem 18** For \(|E| = n\) finite,

\[
\text{ATOM}(\text{FIX}(\neg)) = \begin{cases} 
\{H_k | 0 \leq k \leq (n-1)/2\} & \text{if } n \text{ is odd} \\
\{H_k | 0 \leq k \leq n/2\} & \text{if } n \text{ is even},
\end{cases}
\]

where \(H_k(p) = 1\) iff \(|p| = k\) or \(|p| = n - k\)

Thus for \(|E| = n\), if \(n\) is odd then \(|(\text{FIX}(\neg))_{\text{PI}}| = 2^{(n-1)/2}\); if \(n\) is even \(|(\text{FIX}(\neg))_{\text{PI}}| = 2^{n/2}\). □

The atoms \(H_k\) are expressible as exactly \(k\) or \(n - k\) entities. (Thanks to Marcus Kracht, personal communication, for the basic observation in this theorem).

We end this excursion by pointing out some other entailment patterns using the novel types of NPs we have considered, patterns that we have explored little but which merit further study.

**Pattern 3** The two premisses below jointly entail the conclusion, as indicated. Our interest here is that entailment patterns with proportionality quantifiers are little studied, but in (37) they do entail an existentially quantified sentence of a well understood sort.

\[(37)\] A. More than two thirds of the students passed the exam  
B. At least one third of the students are athletes  
Therefore, at least one student who is an athlete passed the exam

The more general patterns here are determined by relations between the Dets:

(a) More than \(n/m\) of the As are Bs  
(b) At least \(n/m\) of the As are Bs  
At least \(1 - n/m\) of the As are Cs  
Ergo, some A is both a B and a C

More than \(1 - n/m\) of the As are Cs  
Ergo, some A is both a B and a C

**Pattern 4** involves NPs built from Dets of type \((1,1)\), such as those in Transitivity paradigm (39):

\[(39)\] more students than teachers signed the petition  
more teachers than deans signed the petition  
Ergo, more students than deans signed the petition

A first approximation to the more general Transitivity Paradigm is:
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I would like to thank Marcus Kracht for helpful discussion of several of the points made in

Examples of such Dets are more...than..., fewer...than..., exactly as many...as...; elementary arithmetical variants are easy to come by: five more As than Bs have X, seven more Bs than Cs have X, ergo twelve more As than Cs have X:

(41)  n Det A than B X
     m Det B than C X
     Ergo, n+m Det A than C X

And transitivity extends from cardinal comparison to proportionality comparison:

(42)  Proportionately more As than Bs have X
     Proportionately more Bs than Cs have X
     Ergo, proportionately more As than Cs have X

References


Footnotes

1. This article is dedicated to Jim Lambek, the sort of humanitarian and universal scholar universities so desperately need today. While this article does not draw directly on Jim’s work, it is inspired by his mix of linguistic observation and mathematical formulation.

   I would like to thank Marcus Kracht for helpful discussion of several of the points made in
this paper. The research for this work was supported by BSF grant #1999210.

2. Converting the definition to maps from \( n+1 \)-ary relations to \( n \)-ary ones is largely notational:

\[
F(R) = \{x \in E^n | F(\{y \in E | (x,y) \in R\}) = 1\}
\]

3. Some might claim that if a large proportion of the As have B then, a fortiori, a small proportion do as well, interpreting a small proportion to mean at least a small proportion. Such speakers (if there are any) should add Only or Just to the beginning of (19b).

4. Given an S built from three NPs interpreted by GQs F, G, and H there are six functions of the form \( F \circ H \circ G \) varying just by the order of factors. The orders selected will depend on judgments of relative scope. Not studying scope ambiguities or preferences here I have tried to select examples in which our judgment of scope are relatively unequivocal.

5. Thanks to Marcus Kracht (personal communication) for this basic observation.